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Stable trajectory of logistic map

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Abstract In this paper, the stable trajectory of Logistic Map has been investigated by canonical duality theory from the perspective of global optimization. Numerical result of our method shows that it totally differs from traditional chaotic solution solved by Euler method. In addition, we have applied our method to three well-known standard benchmarks in global optimization. Numerical simulations are given to illustrate the effectiveness of the main results.

Keywords Logistic map · Duality · Global optimization

1 Introduction

Chaotic behavior widely exists in a large category of dynamical systems which are highly sensitive to initial conditions (see [1–4]). Small differences in initial conditions, which typically are raised by rounding errors in numerical computation, obtain totally diverging outcomes for chaotic systems even running the same code [5,6]. From the perspective of numerical analysis, discretization error origins from all the continuous functions are approximated in a computer by a finite number of evaluations [7]. In numerical simulation of the ordinary differential equation, discretization error can usually be reduced using a more complicated algorithm with an increasing computational cost, such as the fourth-order Runge–Kutta method. The approximate analytical chaotic solutions of nonlinear differential equations governing the high-dimensional dynamic system can be derived from a differential control method (DCM) [8]. However, each of tiny discretization error involved in each iterative step would be gradually accumulated and finally make significant contributions on chaotic behavior. Estimating model parameters based on chaotic system is becoming challenging due to sensitive dependence to initial conditions. By introducing a smooth function, the authors [9] admits a well-defined maximum, which is equivalent to maximum likelihood estimates. In addition, the Taguchi-sliding-based differential evolution algorithm (TSBDEA) has been developed to solve the problem of system identification for typical chaotic systems, in which a global optimization method combined with the differential evolution algorithm (DEA) is proposed in [10]. Other chaos detection and parameter identification have been discussed by the authors [11] until recently.

In terms of a least-squares minimization procedure, Neuberger and Renka [12] have investigated the difference between chaotic dynamics and chaotic behavior of continuous dynamical system, particularly, the well-known Lorenz attractor. The result indicates that, to some extent, chaotic behavior is purely an artifact

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of numerical roundoff and discretization error by a step-by-step method. Aiming at clarifying the difference between chaotic dynamics and chaotic behavior, we consider the dynamical behaviors of Logistic Map which is described by the following difference equation:

$$x_k = f(x_{k-1}), \quad k = 1, 2, \ldots, n, \tag{1}$$

where x_k is a real number between zero and one, which represents the ratio of existing population to the maximum possible population at year k, and hence x_0 implies the initial ratio of population to maximal population. $f(x) = rx(1-x)$ and r is a positive number, which represents a combined rate for reproduction and starvation. Apparently, discretization error does not exist in difference equation. In [13], the authors introduce randomness in the construction of S-boxes and synthesize substitution boxes by the use of chaotic logistic maps in linear fractional transformation.

Correspondingly, the least-squares minimization of numerical roundoff error can be formulated as follows:

$$P(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{n} \|x_k - r x_{k-1}(1 - x_{k-1})\|^2. \tag{2}$$

We transform dynamical system problem into a least-squares-like problem in terms of global optimization sense. Nonlinear least-squares problem has been discussed for several decades due to its application in science and engineering. For example, an adaptive fuzzy control for a class of discrete-time chaotic systems has been proposed in paper [14], where the parameters of a fuzzy controller law are identified by the least-squares method with dead zone. Meanwhile, canonical duality theory was first developed from Gao and Strang's original work [15] on nonconvex variational problems in large deformation theory, which has been used successfully for solving some interesting nonconvex optimization problems in various disciplines (see, for example, [16–18]). In this paper, we devise an effective solution method based on the canonical duality theory to solve problem (2) and apply it for solving some well-known fourth-order polynomial optimization problems.

The rest of this paper is organized as follows. In the next section, we briefly introduce the canonical duality theory. In Sect. 3, we rewrite the original problem to a new problem, where the decision matrix is expressed in the form of vector. We use the canonical dual transformation to construct the canonical dual problem; the form of analytical solution is obtained from the criticality condition in Sect. 4. Then, we apply our method for solving fourth-order polynomial optimization problems. Finally, some concluding remarks are given in the last section.

2 A brief review of canonical duality theory

Let us consider the following general polynomial optimization problem (primal problem)

$$(\mathcal{P}): \min_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ P(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{f} + W(\boldsymbol{x}) \right\}, \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$ is a given symmetrical indefinite matrix, $\boldsymbol{f} \in \mathbb{R}^n$ is a given vector, and $W(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ is a general nonconvex C^2 function.

In terms of canonical duality theory, we firstly introduce a nonlinear operator (a Gâteaux differentiable geometrical measure)

$$\boldsymbol{\xi} = \Lambda(\boldsymbol{x}) : \mathbb{R}^n \to \mathcal{E}_a \subset \mathbb{R}^m \tag{4}$$

and a convex function $V : \mathcal{E}_a \to \mathbb{R}$ such that $W(\boldsymbol{x})$ can be recast by $W(\boldsymbol{x}) = V(\Lambda(\boldsymbol{x}))$. Then the primal problem can be rewritten as the canonical form:

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ P(\boldsymbol{x}) = V(\Lambda(\boldsymbol{x})) - U(\boldsymbol{x}) \right\}, \tag{5}$$

where $U(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{f}$. The dual variable $\boldsymbol{\varsigma}$ to $\boldsymbol{\xi}$ is defined by the duality mapping

$$\boldsymbol{\varsigma} = \nabla V(\boldsymbol{\xi}) : \mathcal{E}_a \to \mathcal{E}_a^* \subset \mathbb{R}^m, \tag{6}$$

which should be invertible, due to the convexity of $V(\boldsymbol{\xi})$. Then the Legendre conjugate $V^*(\boldsymbol{\varsigma})$ of $V(\boldsymbol{\xi})$ can be uniquely defined by the Legendre transformation

$$V^*(\boldsymbol{\varsigma}) = \text{sta}\{\boldsymbol{\xi}^T\boldsymbol{\varsigma} - V(\boldsymbol{\xi}) | \boldsymbol{\xi} \in \mathcal{E}_a\} \tag{7}$$

and the following canonical duality relations hold on $\mathcal{E}_a \times \mathcal{E}_a^*$:

$$\boldsymbol{\varsigma} = \nabla V(\boldsymbol{\xi}) \Leftrightarrow \boldsymbol{\xi} = \nabla V^*(\boldsymbol{\varsigma}) \Leftrightarrow V(\boldsymbol{\xi}) + V^*(\boldsymbol{\varsigma}) = \boldsymbol{\xi}^T\boldsymbol{\varsigma}. \tag{8}$$

Replacing $W(\boldsymbol{x}) = V(\Lambda(\boldsymbol{x}))$ by $\Lambda(\boldsymbol{x})^T\boldsymbol{\varsigma} - V^*(\boldsymbol{\varsigma})$, we obtain the following total complementary function:

$$\Xi(\boldsymbol{x}, \boldsymbol{\varsigma}) = \Lambda(\boldsymbol{x})^T\boldsymbol{\varsigma} - V^*(\boldsymbol{\varsigma}) - U(\boldsymbol{x}) : \mathbb{R}^n \times \mathcal{E}_a^* \to \mathbb{R}. \tag{9}$$

Using the total complementary function, the canonical dual function $P^d(\varsigma)$ can be formulated as

$$P^d(\varsigma) = \text{sta}\{\Xi(\boldsymbol{x}, \varsigma)|\mathbf{x} \in \mathbb{R}^n\} = U^\Lambda(\varsigma) - V^*(\varsigma), \quad (10)$$

where $U^\Lambda(\varsigma)$ is defined by

$$U^\Lambda(\varsigma) = \text{sta}\{\Lambda(\boldsymbol{x})^T \varsigma - U(\boldsymbol{x})|\boldsymbol{x} \in \mathbb{R}^n\}. \quad (11)$$

Let $\mathcal{S}_a \subset \mathcal{E}_a^*$ be a dual feasible space such that $U^\Lambda(\varsigma)$ is well-defined, and the canonical dual problem can be obtained as

$$(\mathcal{P}^d) : \text{sta}\{P^d(\varsigma)|\varsigma \in \mathcal{S}_a\}. \quad (12)$$

Theorem 1 (Complementary-dual principle)[16] *The problem $(\mathcal{P}^d)$ is canonically dual to the primal problem $(\mathcal{P})$ in the sense that if $(\bar{\boldsymbol{x}}, \bar{\varsigma})$ is a critical point of $\Xi(\boldsymbol{x}, \varsigma)$, then $\bar{\boldsymbol{x}}$ is a feasible solution of $(\mathcal{P})$, $\bar{\varsigma}$ is a feasible solution of $(\mathcal{P}^d)$, and*

$$P(\bar{\boldsymbol{x}}) = \Xi(\bar{\boldsymbol{x}}, \bar{\varsigma}) = P^d(\bar{\varsigma}). \quad (13)$$

In this paper, the geometrical operator $\Lambda(\boldsymbol{x})$ is intrinsically quadratic

$$\Lambda(\boldsymbol{x}) = \{\frac{1}{2}\boldsymbol{x}^T C_k \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{b}_k\} : \mathbb{R}^n \to \mathcal{E}_a \subset \mathbb{R}^m, \quad (14)$$

where $C_k \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b}_k \in \mathbb{R}^n$ are given. In this case, the canonical dual function can be formulated in the form of

$$P^d(\varsigma) = -\frac{1}{2}F^T(\varsigma)G^{-1}(\varsigma)F(\varsigma) - V^*(\varsigma), \quad (15)$$

which is well defined on

$$\mathcal{S}_a = \{\varsigma \in \mathbb{R}^m|F(\varsigma) \in \mathcal{C}_{ol}(G(\varsigma))\}, \quad (16)$$

where $G(\varsigma) = A + \sum_{k=1}^m \varsigma_k C_k$, $F(\varsigma) = \boldsymbol{f} - \sum_{k=1}^m \varsigma_k \boldsymbol{b}_k$, and $\mathcal{C}_{ol}(G(\varsigma))$ denotes the column space of $G(\varsigma)$.

Let the positive domain

$$\mathcal{S}_a^+ = \{\varsigma \in \mathcal{S}_a|G(\varsigma) \succeq 0\}, \quad (17)$$

where $G(\varsigma) \succeq 0$ indicates that $G(\varsigma)$ is a positive semi-definite matrix.

Theorem 2 (Global optimality condition)[16] *Suppose $\bar{\varsigma}$ is a critical point of P^d and $\bar{\boldsymbol{x}} = G^{-1}(\bar{\varsigma})F(\bar{\varsigma})$. If $\bar{\varsigma} \in S_a^+$, then $\bar{\varsigma}$ is a global maximizer of $(\mathcal{P}^d)$ on S_a^+ if and only if $\bar{\boldsymbol{x}}$ is a global minimizer of $(\mathcal{P})$ on $\mathbb{R}^n$, i.e.,*

$$P(\bar{\boldsymbol{x}}) = \min_{\boldsymbol{x}\in\mathbb{R}^n} P(\boldsymbol{x}) \Leftrightarrow \max_{\varsigma\in\mathcal{S}_a^+} P^d(\varsigma) = P^d(\bar{\varsigma}). \quad (18)$$

3 Logistic map

In this section, we discuss about the dynamical behavior of Logistic Map in terms of canonical duality theory from the perspective of global optimization. As usual, one has the perturbed objective function with initial condition x_0:

$$\begin{aligned} P_\epsilon(\boldsymbol{x}) = & \frac{1}{2}\Big\{(x_1 - M)^2 + \sum_{k=1}^{n-1} \epsilon_k \left[r x_k^2 - r x_k + x_{k+1}\right]^2 \\ & - \sum_{k=1}^{n-1} \rho_k x_k^2 - \sum_{k=1}^{n-1} \eta_k x_k\Big\} \\ = & \frac{1}{2}\Big\{\sum_{k=1}^{n-1} \epsilon_k \left[\frac{1}{2}\boldsymbol{x}^T A_k \boldsymbol{x} - B_k^T \boldsymbol{x}\right]^2 \\ & - \frac{1}{2}\boldsymbol{x}^T Q\boldsymbol{x} - D^T\boldsymbol{x} + E\Big\}. \end{aligned} \quad (19)$$

where $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^T$, $Q = diag(2(\rho_1 - 1), 2\rho_2, \ldots, 2\rho_{n-1}, 0)$, $D = [2M + \eta_1, \eta_2, \ldots, \eta_{n-1}, 0]^T$, $E = M^2$, $M = rx_0 - rx_0^2$,

$$\mathbf{A}_k = \begin{pmatrix} 0 & & & & \\ & \cdots & & & \\ & & 2r & \leftarrow k & \\ & & & \cdots & \\ & & & & 0 \end{pmatrix}_{n\times n}, \quad \mathbf{B}_k = \begin{pmatrix} 0 \\ \cdots \\ 0 \\ r \leftarrow k \\ -1 \leftarrow k+1 \\ \cdots \\ 0 \end{pmatrix}_{n\times 1}.$$

Let $\xi_k = \epsilon_k^{\frac{1}{2}}(\frac{1}{2}\boldsymbol{x}^T A_k \boldsymbol{x} - B_k^T \boldsymbol{x}) \quad (k = 1, 2, \ldots, n-1)$, we have

$$V(\boldsymbol{\xi}) = \sum_{k=1}^{n-1} \xi_k^2, \quad (20)$$

$$\varsigma = \partial_\xi V(\xi) = 2\boldsymbol{\xi}, \quad (21)$$

$$V^*(\varsigma) = \boldsymbol{\xi}^T \varsigma - V(\boldsymbol{\xi}) = \frac{1}{4}\varsigma^T \varsigma. \quad (22)$$

According to (6)–(9), the total complementary function can be defined as

$$\begin{aligned} \Xi(\boldsymbol{x}, \varsigma) = & \sum_{k=1}^{n-1} \epsilon_k^{\frac{1}{2}} (\frac{1}{2}\boldsymbol{x}^T A_k \boldsymbol{x} - B_k^T \boldsymbol{x})\varsigma_k - \frac{1}{4}\sum_{k=1}^{n-1} \varsigma_k^2 \\ & - \frac{1}{2}\boldsymbol{x}^T Q\boldsymbol{x} - D^T\boldsymbol{x} + E \\ = & \frac{1}{2}\boldsymbol{x}^T \left[\sum_{k=1}^{n-1} \epsilon_k^{\frac{1}{2}} A_k \varsigma_k - Q\right]\boldsymbol{x} \end{aligned}$$

$$-\left[\sum_{k=1}^{n-1}\epsilon_k^{\frac{1}{2}}B_k^T\varsigma_k + D^T\right]\boldsymbol{x}+E-\frac{1}{4}\sum_{k=1}^{n-1}\varsigma_k^2. \tag{23}$$

From the critical point of $\nabla\Xi(\boldsymbol{x},\boldsymbol{\varsigma})$,

$$\boldsymbol{x} = G^{-1}(\boldsymbol{\varsigma})F(\boldsymbol{\varsigma}), \tag{24}$$

where

$$G(\boldsymbol{\varsigma}) = \sum_{k=1}^{n-1}\epsilon_k^{\frac{1}{2}}A_k\varsigma_k - Q, \tag{25}$$

$$F(\boldsymbol{\varsigma}) = \sum_{k=1}^{n-1}\epsilon_k^{\frac{1}{2}}B_k^T\varsigma_k + D^T. \tag{26}$$

Substituting (24) into the total complementary function $\Xi(\boldsymbol{x},\boldsymbol{\varsigma})$, each of canonical dual problem can be formalized

$$\phi_1(\boldsymbol{\varsigma}) = -\frac{1}{4}\left\{\frac{(2M+\eta_1+a\epsilon_1^{\frac{1}{2}}\varsigma_1)^2}{a\epsilon_1^{\frac{1}{2}}\varsigma_1-\rho_1+1}+\varsigma_1^2\right\}, \tag{27}$$

$$\phi_{n-2}(\boldsymbol{\varsigma}) = -\frac{1}{4}\left\{\frac{(\eta_{n-2}+a\epsilon_{n-2}^{\frac{1}{2}}\varsigma_{n-2}-\epsilon_{n-3}^{\frac{1}{2}}\varsigma_{n-3})^2}{a\epsilon_{n-2}^{\frac{1}{2}}\varsigma_{n-2}-\rho_{n-2}}+\varsigma_{n-2}^2\right\}, \tag{28}$$

$$\phi_{n-1}(\boldsymbol{\varsigma}) = \frac{1}{4}\left\{\frac{(\eta_{n-1}-\epsilon_{n-2}^{\frac{1}{2}}\varsigma_{n-2})^2}{\rho_{n-1}}\right\}+E. \tag{29}$$

The canonical dual problem can be finally formulated as

$$P^d(\boldsymbol{\varsigma}) =: \text{sta}\left\{\sum_{k=1}^{n-1}\phi_k(\boldsymbol{\varsigma})\right\}, \tag{30}$$

for any $\varsigma_k \in \mathcal{S}_a$.

Example 1 Consider logistic map with $n = 200$ as the following least-squares minimization problem

$$P(\boldsymbol{x}) = \frac{1}{2}\sum_{k=1}^{200}\|x_k - rx_{k-1}(1-x_{k-1})\|^2, \tag{31}$$

where $x_0 = 0.15$ and $r = 4$. Let $\epsilon_k = 1\times10^{-6}$, $\rho_k = 1\times10^{-6}$, and $\eta_k = 0$, one has the global minimum $P(\boldsymbol{x}) = 1.491402\times10^{-7}$ solved by L-BFGS-B method [19]. Compared to the traditional iterative method, the trajectory of global optimization method turns out to be a stable path convergence to the equilibrium of Logistic map, see details in Fig. 1.

4 Applications

In this section, we will discuss three problems related with the proposed approach for Logistic Map. All these problems can be regarded as nonlinear least-squares functions, which are well known as standard benchmarks for global optimization algorithms as well as heuristic methods.

4.1 Rosenbrock function

First of all, consider the standard Rosenbrock Function problem

$$f(\boldsymbol{x}) = \sum_{i=1}^{n}\left[(x_i-1)^2+100(x_{i+1}-x_i^2)^2\right]. \tag{32}$$

Mathematically speaking, the Rosenbrock function is a nonconvex function and called Rosenbrock's banana, in which the global minimum is inside a long, narrow, parabolic shaped flat valley. For small n, the polynomials optimization problem can be determined exactly as well as the number of real roots in terms of Sturm's theorem, while all roots can be bounded in the $|x_i| \le 2.4$ [20]. However, the case n changes into a large scale, for example $n > 1{,}000$, this method breaks down due to the large size of the coefficients involved. Fortunately, this problem can be tackled analytically by canonical dual theory and the global solution can be obtained within given tolerance.

Let $\boldsymbol{\xi} = \Lambda(\boldsymbol{x})$, then

$$\boldsymbol{\xi} = \{\xi_k\} = \epsilon_k^{\frac{1}{2}}(x_k^2 - x_{k+1}) \in \xi_a \subset \mathbb{R}^{n-1}, \tag{33}$$

$$V(\boldsymbol{\xi}) = 100\sum_{k=1}^{n-1}\xi_k^2, \tag{34}$$

and the duality relation

$$\boldsymbol{\varsigma} = \{\varsigma_k\} = \left\{\frac{\partial V(\boldsymbol{\xi})}{\partial\xi_k}\right\} = \{200\xi_k\}. \tag{35}$$

Thus,

$$\xi_k = \frac{1}{200}\varsigma_k. \tag{36}$$

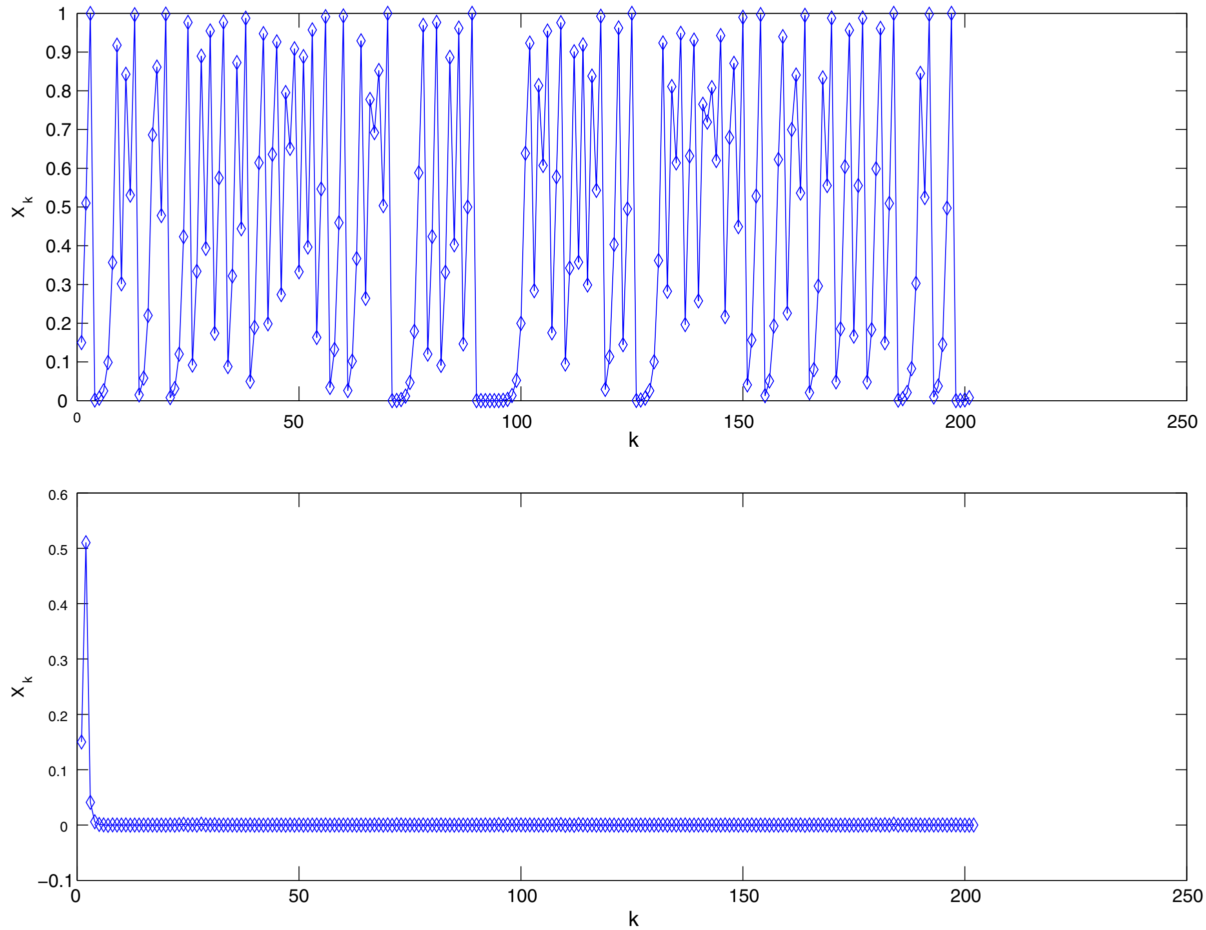

Fig. 1 Chaotic trajectory versus stable trajectory (200 iterations)

We have

$$V_k^*(\varsigma_k) = \xi_k \varsigma_k - V_k(\xi_k) = \xi_k \varsigma_k - 100\xi_k^2 = \frac{1}{400}\varsigma_k^2. \tag{37}$$

According to (6)–(9), the total complementary function can be defined as

$$\Xi(\boldsymbol{x}, \boldsymbol{\varsigma}) = \sum_{k=1}^{n-1}(x_i - 1)^2 + \Lambda(\boldsymbol{x})^T \boldsymbol{\varsigma} - V^*(\boldsymbol{\varsigma}) = \sum_{k=1}^{n-1}\left[(x_k - 1)^2 + \epsilon_k^{\frac{1}{2}}(x_k^2 - x_{k+1})\varsigma_k - \frac{1}{400}\varsigma_k^2\right], \tag{38}$$

with the canonical dual feasible space $\mathcal{S}_a \subset \mathbb{R}^{n-1}$ defined by

$$\mathcal{S}_a = \left\{\boldsymbol{\varsigma} \in \mathcal{S} \mid \epsilon_k^{\frac{1}{2}}\varsigma_k + 1 \neq 0, \ \forall k = 1, \ldots, n-2, \ \varsigma_{n-1} = 0\right\}. \tag{39}$$

For a fixed ς, the criticality condition $\nabla_{\boldsymbol{x}} \Xi(\boldsymbol{x}, \boldsymbol{\varsigma}) = 0$ leads to the solution of the canonical equilibrium equation, which can be uniquely determined as

$$x_k = \frac{\epsilon_{k-1}^{\frac{1}{2}}\varsigma_{k-1} + 2}{2(\epsilon_k^{\frac{1}{2}}\varsigma_k + 1)}. \tag{40}$$

Substituting this result (40) into the total complementary function $\Xi(\boldsymbol{x}, \boldsymbol{\varsigma})$, the canonical dual problem can be finally formulated as

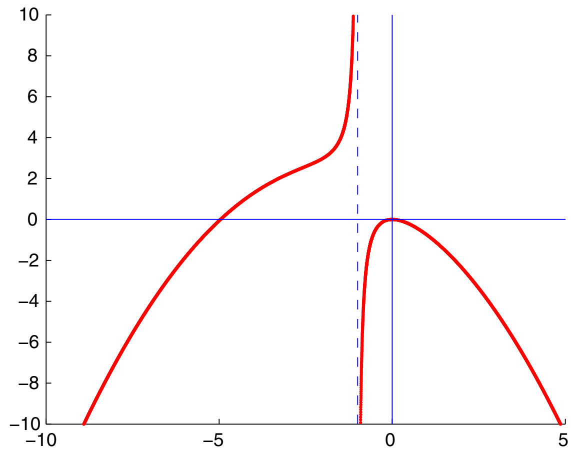

Fig. 2 The portrayal of dual problem with respect to ς_1

$$P^d(\boldsymbol{\varsigma}) =: \text{sta}\left\{n-1-\sum_{k=1}^{n-1}\left[\frac{(\epsilon_{k-1}^{\frac{1}{2}}\varsigma_{k-1}+2)^2}{4(\epsilon_k^{\frac{1}{2}}\varsigma_k+1)}+\frac{1}{400}\varsigma_k^2\right]\right\}, \tag{41}$$

for any $\varsigma_k \in \mathcal{S}_a$.

Example 2 Consider the typical Rosenbrock function with $n = 3$ as the following

$$f(\boldsymbol{x}) = (x_1-1)^2 + 100(x_2-x_1^2)^2 + (x_2-1)^2 + 100(x_3-x_2^2)^2.$$

Correspondingly, its dual problem is given

$$P^d(\boldsymbol{\varsigma}) =: \text{sta}\left\{2-\left[\frac{1}{(\varsigma_1+1)}+\frac{1}{400}\varsigma_1^2+\frac{(\varsigma_1+2)^2}{4}\right]\right\},$$

where $\varsigma_1 = 200(x_1^2 - x_2)$ and $\varsigma_2 = 200(x_2^2 - x_3)$. In this case, the dual problem has a unique critical point $\boldsymbol{\varsigma} = (0, 0)$ in this space

$$\mathcal{S}_a^+ = \left\{\boldsymbol{\varsigma} \in \mathcal{S} \mid \epsilon_1^{\frac{1}{2}}\varsigma_1 \geq -1, \quad \varsigma_2 = 0\right\},$$

which coincides with the graph of dual function, see details in Fig. 2.

Therefore, x of primal function is given by (40)

$$\boldsymbol{x}^* = \left[\frac{1}{\varsigma_1+1}, \frac{\varsigma_1+2}{2(\varsigma_2+1)}, x_2^2\right] = [1, 1, 1],$$

is a global minimization.

Table 1 Numerical results of Rosenbrock function with different n

n	$P(\boldsymbol{x}^*)$	$P^d(\boldsymbol{\varsigma}^*)$
4,000	3.338215e−10	1.41049e−10
6,000	7.544455e−09	1.662183e−09
8,000	5.249997e−10	2.340217e−10
1,0000	3.169148e−09	6.554447e−10

Example 3 Consider the large-scale Rosenbrock functions with n= 4,000, 6,000, 8,000, and 10,000. Similarly, by employing L-BFGS-B method and randomly generating initial values on the interval $(-1, 10]$, the numerical results are given in Table 1.

4.2 Dixon and Price function

Consider Dixon and Price function

$$f(\boldsymbol{x}) = (x_1-1)^2 + \sum_{i=2}^{n} i(2x_i^2 - x_{i-1})^2, \tag{42}$$

which is a fourth-order polynomial optimization problem without constraint. The global minimum of this problem is $x_i = 2^{\frac{\lambda-1}{\lambda}}(\lambda = 2^{i-1})$ and $f(\boldsymbol{x}^*) = 0$ which means $\lim_{i\to\infty} x_i = \frac{1}{2}$. In fact, there are two global minimizers that can be obtained analytically by canonical dual theory.

Consider the perturbed function as follows

$$\begin{aligned} P_\epsilon(\boldsymbol{x}) &= (x_1-1)^2 + \sum_{k=2}^{n} k\epsilon_k(2x_k^2 - x_{k-1})^2 - \sum_{k=1}^{n}\rho_k x_k^2 - \sum_{k=1}^{n}\eta_k x_k \\ &= \sum_{k=2}^{n} k\epsilon_k\left(\frac{1}{2}\boldsymbol{x}^T A_k\boldsymbol{x} - B_k^T\boldsymbol{x}\right)^2 - \frac{1}{2}\boldsymbol{x}^T Q\boldsymbol{x} - D^T\boldsymbol{x} + E, \end{aligned} \tag{43}$$

where $A_k = diag\{0, \ldots, 4, \ldots 0\}$; $B_k = [1, 0, \ldots, 0]^T$; $Q = diag\{2(\rho_1 - 1), 2\rho_2, \ldots\}$; $D = diag\{2 + \eta_1, \eta_2, \ldots\}$; and $E = 1$.

Let $\xi_k = (k\epsilon_k)^{\frac{1}{2}}(\frac{1}{2}\boldsymbol{x}^T A_k\boldsymbol{x} - B_k^T\boldsymbol{x})(k = 2, 3, \ldots, n)$, we have

$$V(\boldsymbol{\xi}) = \sum_{k=2}^{n}\xi_k^2, \tag{44}$$

$$\boldsymbol{\varsigma} = \partial_{\boldsymbol{\xi}} V(\boldsymbol{\xi}) = 2\boldsymbol{\xi}, \tag{45}$$

$$V^*(\boldsymbol{\varsigma}) = \boldsymbol{\xi}^T \boldsymbol{\varsigma} - V(\boldsymbol{\xi}) = \frac{1}{4}\boldsymbol{\varsigma}^T \boldsymbol{\varsigma}. \tag{46}$$

According to (6)–(9), the total complementary function can be defined as

$$\begin{aligned}\Xi(\boldsymbol{x}, \boldsymbol{\varsigma}) &= \sum_{k=2}^{n} (k\epsilon_k)^{\frac{1}{2}} \left(\frac{1}{2}\boldsymbol{x}^T A_k \boldsymbol{x} - B_k^T \boldsymbol{x}\right)\varsigma_k \\ &\quad -\frac{1}{4}\sum_{k=2} n\varsigma_k^2 - \frac{1}{2}\boldsymbol{x}^T Q\boldsymbol{x} - D^T\boldsymbol{x} + E \\ &= \frac{1}{2}\boldsymbol{x}^T \left[\sum_{k=2}^{n} (k\epsilon_k)^{\frac{1}{2}} A_k \varsigma_k - Q\right]\boldsymbol{x} \\ &\quad + \left[\sum_{k=2}^{n} (k\epsilon_k)^{\frac{1}{2}} B_k^T \varsigma_k + D^T\right]\boldsymbol{x} + E \\ &\quad -\frac{1}{4}\sum_{k=2}^{n} \varsigma_k^2.\end{aligned} \tag{47}$$

From the critical point of $\nabla\Xi(\boldsymbol{x}, \boldsymbol{\varsigma})$,

$$\boldsymbol{x} = G^{-1}(\boldsymbol{\varsigma})F(\boldsymbol{\varsigma}), \tag{48}$$

where

$$G(\boldsymbol{\varsigma}) = \sum_{k=2}^{n} (k\epsilon_k)^{\frac{1}{2}} A_k \varsigma_k - Q, \tag{49}$$

$$F(\boldsymbol{\varsigma}) = \sum_{k=2}^{n} (k\epsilon_k)^{\frac{1}{2}} B_k^T \varsigma_k + D^T. \tag{50}$$

Substituting (48) into the total complementary function $\Xi(\boldsymbol{x}, \boldsymbol{\varsigma})$, each of the canonical dual problem can be formalized

$$\phi_1(\boldsymbol{\varsigma}) = -\frac{1}{2}\frac{\left[2 + \eta_1 + (2\epsilon_2)^{\frac{1}{2}}\varsigma_2\right]^2}{2(1-\rho_1)} + 1, \tag{51}$$

$$\begin{aligned}\phi_{n-1}(\boldsymbol{\varsigma}) &= -\frac{1}{2}\frac{\left[\eta_{n-1} + (n\epsilon_n)^{\frac{1}{2}}\varsigma_n\right]^2}{4((n-1)\epsilon_{n-1})^{\frac{1}{2}}\varsigma_{n-1} - 2\rho_{n-1}} \\ &\quad -\frac{1}{4}\varsigma_{n-1}^2,\end{aligned} \tag{52}$$

$$\phi_n(\boldsymbol{\varsigma}) = -\frac{1}{2}\frac{\eta_n^2}{4(n\epsilon_n)^{\frac{1}{2}}\varsigma_n - 2\rho_n} - \frac{1}{4}\varsigma_n^2. \tag{53}$$

The canonical dual problem can be eventually formulated as

$$P^d(\boldsymbol{\varsigma}) =: \text{sta}\left\{\sum_{k=1}^{n} \phi_k(\boldsymbol{\varsigma})\right\}, \tag{54}$$

for any $\varsigma_k \in \mathcal{S}_a$.

Example 4 Consider the standard test function with $n = 2$ as the following

$$f(\boldsymbol{x}) = (x_1 - 1)^2 + 2(2x_2^2 - x_1)^2,$$

which is a large valley.

The corresponding dual function is given by

$$P^d(\varsigma) =: \text{sta}\left\{1 - \frac{1}{4}(2 + 2^{\frac{1}{2}}\varsigma)^2\right\},$$

where $\varsigma = 2^{\frac{1}{2}}(2x_2^2 - x_1)$. From the aforementioned approach, the critical point $\varsigma = 0$ in this case. Hence, the global minimums are given by

$$\begin{aligned}\boldsymbol{x}^* &= \left[1 + \varsigma, \pm\sqrt{\frac{x_1}{2}}\right] \\ &= \left[1, \pm\frac{\sqrt{2}}{2}\right].\end{aligned}$$

Example 5 In this example, we discuss the case $n = 3{,}000$. Likewise, by employing L-BFGS-B method and randomly generating initial values on the interval [0, 10], numerical results of dual problem $P^d(\boldsymbol{\varsigma}^*)$= 7.0720e−6 are obtained. However, the results of the primal problem by dual transformation $P(\boldsymbol{x}^*) = 0.6778$ are unacceptable because the computation errors caused by perturbation are accumulated into an undeniable scale when n increases into a large size. Thus, these results of the primal problem should be regarded as new start points to be refined by gradient-based optimization method. Finally, we have the refined $P(\boldsymbol{x}^*)$ = 1.2704e−6. Remark that this procedure can be iteratively repeated until the results are achieved at high accuracy.

4.3 Powell function

Consider Powell function problem

$$\begin{aligned}f(\boldsymbol{x}) = \sum_{i=1}^{n/4} &[(x_{4i-3} + 10x_{4i-2})^2 + 5(x_{4i-1} - x_{4i})^2 \\ &+ (x_{4i-2} - x_{4i-1})^4 + 10(x_{4i-3} - x_{4i})^4],\end{aligned} \tag{55}$$

which was introduced in 1962 by Powell as a singular unconstrained optimization problem. Powell function is a type of optimization problem which can be split into small size with respect to n. For each small independent problem, standard starting point of numerical test is $\boldsymbol{x}_0 = (3, -1, 0, 1)^T$ and the Hessian matrix at global minimum is bi-singular. To solve it, we utilize proximal point method to tackle it.

Let $\xi_k = (x_{4k-2} - x_{4k-1})^2$ and $\varepsilon_k = (x_{4k-3} - x_{4k})^2$, we have

$$V(\boldsymbol{\xi}) = \sum_{k=1}^{n/4} \xi_k^2, \tag{56}$$

$$\boldsymbol{\varsigma} = \partial_{\boldsymbol{\xi}} V(\boldsymbol{\xi}) = 2\boldsymbol{\xi}, \tag{57}$$

$$V^*(\boldsymbol{\varsigma}) = \boldsymbol{\xi}^T \boldsymbol{\varsigma} - V(\boldsymbol{\xi}) = \frac{1}{4} \boldsymbol{\varsigma}^T \boldsymbol{\varsigma}. \tag{58}$$

$$V(\boldsymbol{\varepsilon}) = 10 \sum_{k=1}^{n/4} \varepsilon_k^2, \tag{59}$$

$$\boldsymbol{\sigma} = \partial_{\boldsymbol{\varepsilon}} V(\boldsymbol{\varepsilon}) = 20\boldsymbol{\varepsilon}, \tag{60}$$

$$V^*(\boldsymbol{\sigma}) = \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} - V(\boldsymbol{\varepsilon}) = \frac{1}{40} \boldsymbol{\sigma}^T \boldsymbol{\sigma}. \tag{61}$$

The total complementary function can be defined as

$$\begin{aligned}
\Xi(\boldsymbol{x}, \boldsymbol{\varsigma}, \boldsymbol{\sigma})
&= \sum_{k=1}^{n/4} \Big[(x_{4k-3} + 10x_{4k-2})^2 + 5(x_{4k-1} - x_{4k})^2 \\
&\quad + (x_{4k-2} - x_{4k-1})^2 \varsigma_k + 10(x_{4k-3} - x_{4k})^2 \sigma_k \\
&\quad - \frac{1}{4}\varsigma_k^2 - \frac{1}{40}\sigma_k^2 + \rho_k \|\boldsymbol{x} - \boldsymbol{x}^*\|^2 \Big] \\
&= \sum_{k=1}^{n/4} [\boldsymbol{x}^T G_k \boldsymbol{x} - 2F_k^T \boldsymbol{x} - \frac{1}{4}\varsigma_k^2 - \frac{1}{40}\sigma_k^2 + \rho_k (\boldsymbol{x}^*)^2],
\end{aligned} \tag{62}$$

where $\boldsymbol{x} = [x_{4k-3}, x_{4k-2}, x_{4k-1}, x_{4k}]^T$ is variable vector, ρ_k is a regularized parameter, and $\boldsymbol{x}^*$ is the start point for every small independent problem in each step (see details in [21]),

$$G_k = \begin{bmatrix} (1+10\sigma_k+\rho_k) & 10 & 0 & -10\sigma_k \\ 10 & (100+\varsigma_k+\rho_k) & -\varsigma_k & 0 \\ 0 & -\varsigma_k & (5+\varsigma_k+\rho_k) & -5 \\ -10\sigma_k & 0 & -5 & (5+10\sigma_k+\rho_k) \end{bmatrix}$$

and

$$F_k = \begin{bmatrix} -\rho_k \\ -\rho_k \\ -\rho_k \\ -\rho_k \end{bmatrix}.$$

From the critical point of $\nabla \Xi(\boldsymbol{x}, \boldsymbol{\varsigma}, \boldsymbol{\sigma})$,

$$\boldsymbol{x} = G_k^{-1} F_k. \tag{63}$$

The canonical dual problem of Powell singular function can be formulated as

$$P^d(\boldsymbol{\varsigma}, \boldsymbol{\sigma}) =: \text{sta} \sum_{k=1}^{n} \left\{ -F_k^T G_k^{-1} F_k - \frac{1}{4}\varsigma_k^2 - \frac{1}{40}\sigma_k^2 \right\}, \tag{64}$$

for any $\varsigma_k \in \mathcal{S}_a$. Note that $\rho_k(\boldsymbol{x}^*)^2$ is constant and omitted here.

Example 6 Consider the standard test function with $n = 4$ as the following

$$\begin{aligned} f(x) = (x_4 + 10x_3)^2 + 5(x_2 - x_1)^2 + (x_3 - x_2)^4 \\ + 10(x_4 - x_1)^4. \end{aligned} \tag{65}$$

Let $\rho = 1 \times 10^{-4}$. The global minimizer can be obtained immediately by the algorithm in [21], which is $\boldsymbol{x}^* = 1 \times 10^{-8} \times [0.0650 \ \ -0.0065 \ \ 0.1534 \ \ 0.1534]$. All the experiments were run on a HP Pavilion G6 computer with an Intel(R) Core(TM)I5-2430M 2.4GHz processor and 4,00 GB of memory.

5 Conclusion

A new numerical method for dynamical system has been discussed from the perspective of global optimization in this paper. Compared to the chaotic behavior of logistic map, a stable trajectory has been obtained in terms of canonical dual theory. Some chaos from traditional numerical iteration of dynamical system, to some extent, are artificial results caused by computation error, which should be differentiated by chaos in physics. Besides, three least-squares-like functions have been investigated by the proposed method. For small size, the global minimizer can be obtained analytically. Numerical simulations show that our method can achieve good performance even in a large-scale problem. In further research, we will try to investigate chaotic phenomena governed by differential equation and determine the effect of computational errors.

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